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Instantons and the 5D $U(1)$ gauge theory with extra adjoint

Rubik Poghossian[1] **and Marine Samsonyan**[2]

[1] Yerevan Physics Institute, Alikhanian Br. St. 2, 0036 Yerevan, Armenia
[2] Dipartimento di Fisica, Universitá di Roma 'Tor Vergata' INFN Sezione di Roma II, Via della Ricerca Scientifica, 00133 Roma, Italy

E-mail: poghos@yerphi.am and marine.samsonian@roma2.infn.it

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Abstract
In this paper, we compute the partition function of 5D supersymmetric $U(1)$ gauge theory with extra adjoint matter in general Ω background. It is well known that such partition functions encode very rich topological information. We show in particular that unlike the case with no extra matter, the partition function with extra adjoint at some special values of the parameters directly reproduces the generating function for the Poincare polynomial of the moduli space of instantons. We compare our results with those recently obtained by Iqbal *et al* (Refined topological vertex, cylindric partitions and the $U(1)$ adjoint theory, arXiv:0803.2260), who used the so-called refined topological vertex method.

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1. Introduction

Recent progress in understanding non-perturbative phenomena in supersymmetric Yang–Mills theories due to direct multi-instanton calculations is quite impressive. Two main ideas played an essential role in all these developments. First was the realization that the supersymmetric Yang–Mills action induced to the moduli space of instantons can be represented as an integral over a closed, equivariant with respect to the diagonal part of the gauge group form [1]. This observation leads to a crucial simplification reducing the SYM path integral to an integral over the sub-manifold of the moduli space of instantons which is stable with respect to the diagonal part of the gauge group. The next brilliant idea, which became the corner stone of all further developments, was suggested by Nekrasov in [2]. The idea was to generalize the theory involving into the game in equal footing with the already mentioned global diagonal gauge transformations also the diagonal part of the (Euclidean) space-time rotations. This is so crucial because the subset of the instanton moduli space invariant under this combined group action consists only of a finite number of points.

In the case of the gauge group $U(N)$, this fixed point set is in one-to-one correspondence with the set of array of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_N)$ with total number of boxes $|\vec{Y}|$ being equal to the instanton charge k. Thus, to calculate the path integral for the various 'protected' by super-symmetry physical quantities, one needs to know only the pattern of how the combined group acts in the neighborhoods of the fixed point. All this information can be encoded in the character of the group action in the tangent space at given fixed points. An elegant formula for this character which played a significant role in both physical and mathematical applications was proposed in [3] (see equation (2.1)). Let us note at once that combining space-time rotations with the gauge transformations besides giving a huge computational advantage due to the finiteness of the fixed point set has also a major physical significance generalizing the theory to the case with certain non-trivial graviphoton backgrounds [2]. In order to recover the standard flat space quantities (say the Seiberg–Witten prepotential of $\mathcal{N} = 2$ super-Yang–Mills theory), one should take the limit when the space-time rotation angles vanish. It is shown by Nekrasov and Okounkov [4] that in this limit the sum over the arrays of Young diagrams is dominated by a single array with specific 'limiting shape'. As a result, it becomes possible to handle the entire instanton sum expressing all relevant quantities in terms of emerging Seiberg–Witten curve [5]. Note that only the entire sum but not its truncated part exhibits remarkable modular properties which allows one to investigate the rich phase structure of SYM theories. This is why all attempts to investigate the entire instanton sum also in a general case (i.e. keeping finite the space-time rotation angles) seem quite natural. Unfortunately, there was little progress till now in this direction besides the simplest case of the gauge group $U(1)$. Though the $U(1)$ 4D theory in flat background is trivial, the general 5D $U(1)$ theories compactified on a circle[3] being rather non-trivial nevertheless in many cases admit full solution. In what follows, we investigate the partition function of 5D gauge theory with an extra adjoint hypermultiplet. It is not surprising that such partition functions encode very rich topological information. As a manifestation we argue that unlike the case with no extra matter, at some special values of the parameters this partition function directly reproduces the generating function of the Poincare polynomial for the moduli space of instantons. We check this conclusion explicitly computing the partition function in the case of the gauge group $U(1)$. We compare our result with that recently obtained by Iqbal *et al* [6] who used the refined topological vertex method [12] to find the same partition function and present our comments on discrepancies we found. See [13] for an earlier attempt to construct a refined version of the topological vertex.

2. The $U(1)$ theory with adjoint matter

The weight decomposition of the torus action on the tangent space at the fixed point $\vec{Y} = (Y_1, \ldots, Y_N)$ is given by [3]

$$\chi = \sum_{\alpha,\beta=1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s\in Y_\alpha} \left(T_1^{-l_\beta(s)} T_2^{a_\alpha(s)+1}\right) + \sum_{s\in Y_\beta} \left(T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)}\right) \right\}, \tag{2.1}$$

where $e_1, \ldots, e_N$ are the elements of (complexified) maximal torus of the gauge group $U(N)$, T_1, T_2 belong to the maximal torus of the (Euclidean) space-time rotations and $a_\alpha(s)$

[3] Roughly speaking the main technical difference between 4D and 5D cases is that in the former case the above-mentioned combined group enters into the game in the infinitesimal level while in the latter case the main roles are played by the finite group elements.

$(l_\alpha(s))$ measures the distance from the location of the box s to the edge of the Young diagram Y_α in the vertical (horizontal) direction.

The 5D partition function in the pure $\mathcal{N} = 2$ theory could be read off from the above character, formula (2.1)

$$Z = \sum_{\vec{Y}} \frac{\mathbf{q}^{|\vec{Y}|}}{\prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \left(1 - e_\beta e_\alpha^{-1} T_1^{-l_\beta(s)} T_2^{a_\alpha(s)+1}\right) \prod_{s\in Y_\beta} \left(1 - e_\beta e_\alpha^{-1} T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)}\right)} \tag{2.2}$$

From the mathematical point of view this quantity could be regarded as the character of the torus action on the space of holomorphic functions of the moduli space of instantons. The Nekrasov's partition function for 4D theory could be obtained tuning the parameters $\mathbf{q} \to \beta^{2N}\mathbf{q}$, $T_1 \to \exp -\beta\epsilon_1$, $T_2 \to \exp -\beta\epsilon_2$, $e_\alpha \to -\beta v_\alpha$ and tending $\beta \to 0$, where $v_1, \ldots, v_N$ are the expectation values of the chiral superfield and ϵ_1, ϵ_2 characterize the strength of the graviphoton background (sometimes called the Ω background).

Fortunately, instanton counting is powerful enough to handle also the cases when an extra hypermultipet in adjoint or several fundamental hypermultiplets are present. In the case with an adjoint hypermultiplet instead of (2.1), one starts with the (super) character [7]

$$\chi = (1 - T_m) \sum_{\alpha,\beta=1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s\in Y_\alpha} \left(T_1^{-l_\beta(s)} T_2^{a_\alpha(s)+1}\right) + \sum_{s\in Y_\beta} \left(T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)}\right) \right\}. \tag{2.3}$$

One way to interpret this character is to imagine that each (complex) 1d eigenspace of the torus action is complemented by a grassmanian eigenspace with exactly the same eigenvalues of the torus action. In addition, an extra $U(1)$ action is introduced so that $T_m \in U(1)$ acts trivially on bosonic directions while on each grassmanian coordinate it acts in its fundamental representation. Then (2.3) is the super-trace of the extended torus action on the super-tangent space at given fixed point. The corresponding 5D partition function now reads

$$Z = \sum_{\vec{Y}} \mathbf{q}^{|Y|} \prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \frac{\left(1 - T_m e_\beta e_\alpha^{-1} T_1^{-l_\beta(s)} T_2^{a_\alpha(s)+1}\right)}{\left(1 - e_\beta e_\alpha^{-1} T_1^{-l_\beta(s)} T_2^{a_\alpha(s)+1}\right)} \prod_{s\in Y_\beta} \frac{\left(1 - T_m e_\beta e_\alpha^{-1} T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)}\right)}{\left(1 - e_\beta e_\alpha^{-1} T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)}\right)} \tag{2.4}$$

Each term here could be thought of as a trace over the space of local holomorphic forms, with parameter T_m counting the degrees of forms. Hence, the sum over the fixed points is expected to give the super-trace over the globally defined holomorphic forms. We see that Z_{adj} is an extremely rich quantity from both physical and mathematical points of view. It is interesting to note that at special values of the parameters Z_{adj} directly reproduces the generating function for the Poincare polynomial of the moduli space of $U(N)$ instantons. Indeed, following [8] part (3.3) let us assume that $T_2 \gg T_{a_1} > \cdots > T_{a_N} \gg T_1 > 0$. It is easy to see that in the limit when all these parameters go to zero each fraction under the products in (2.4) tends to T_m or 1 depending on whether we have a negative weight direction or not (see the classification of negative directions in [8], proof of corollary 3.10). We will see this explicitly in the simplest case $N = 1$ when the moduli space of instantons coincides with the Hilbert scheme of points on $\mathbb{C}^2$.

From now on we will restrict ourselves to the simplest case of the $U(1)$ gauge group, when the partition function could be computed in a closed way. The partition function of the pure $\mathcal{N} = 2$, $U(1)$ theory has the form [9]

$$Z = \sum_{Y} \frac{\mathbf{q}^{|Y|}}{\prod_{s\in Y} \left(1 - T_1^{-l(s)} T_2^{a(s)+1}\right)\left(1 - T_1^{l(s)+1} T_2^{-a(s)}\right)}$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{\mathbf{q}^n}{n\left(1-T_1^n\right)\left(1-T_2^n\right)}\right). \tag{2.5}$$

This remarkable combinatorial identity in the 4D limit and in the 'self dual' case $\epsilon_1 = -\epsilon_2$ boils down to the Burnside's theorem

$$\sum_{|\lambda|=n}(\dim \mathbf{R}_\lambda)^2 = n!, \tag{2.6}$$

where $\mathbf{R}_\lambda$ is the irreducible representation of the symmetric group given by the Young diagram λ.

Now let us turn to the $U(1)$ theory with adjoint matter. Doing low instanton calculations using (2.4) is straightforward and gives

$$\begin{aligned}\log Z_{\text{adj}} = {} & \frac{\mathbf{q}\left(1+T_m\mathbf{q}+T_m^2\mathbf{q}^2+T_m^3\mathbf{q}^3\right)(1-T_mT_1)(1-T_mT_2)}{(1-T_1)(1-T_2)} \\ & + \frac{\mathbf{q}^2\left(1+T_m^2\mathbf{q}^2\right)\left(1-T_m^2T_1^2\right)\left(1-T_m^2T_2^2\right)}{2\left(1-T_1^2\right)\left(1-T_2^2\right)} + \frac{\mathbf{q}^3\left(1-T_m^3T_1^3\right)\left(1-T_m^3T_2^3\right)}{3\left(1-T_1^3\right)\left(1-T_2^3\right)} \\ & + \frac{\mathbf{q}^4\left(1-T_m^4T_1^4\right)\left(1-T_m^4T_2^4\right)}{4\left(1-T_1^4\right)\left(1-T_2^4\right)} + O(\mathbf{q}^4).\end{aligned} \tag{2.7}$$

These drove us to the conjecture that the exact formula is

$$\log Z_{\text{adj}} = \sum_{n=1}^{\infty}\frac{\mathbf{q}^n\left(1-T_m^nT_1^n\right)\left(1-T_m^nT_2^n\right)}{n\left(1-T_1^n\right)\left(1-T_2^n\right)\left(1-T_m^n\mathbf{q}^n\right)}, \tag{2.8}$$

which is equivalent to the following highly non-trivial combinatorial identity:

$$\begin{aligned}Z_{\text{adj}} &= \sum_Y \mathbf{q}^{|Y|}\prod_{s\in Y}\frac{\left(1-T_mT_1^{-l(s)}T_2^{a(s)+1}\right)\left(1-T_mT_1^{l(s)+1}T_2^{-a(s)}\right)}{\left(1-T_1^{-l(s)}T_2^{a(s)+1}\right)\left(1-T_1^{l(s)+1}T_2^{-a(s)}\right)} \\ &= \exp\left(\sum_{n=1}^{\infty}\frac{\mathbf{q}^n(1-(T_mT_1)^n)(1-(T_mT_2)^n)}{n\left(1-T_1^n\right)\left(1-T_2^n\right)(1-(T_m\mathbf{q})^n)}\right).\end{aligned} \tag{2.9}$$

Indeed, calculations with Mathematica code up to 10 instantons further convinced us that this formula is indeed correct. Note that the 4D limit of this identity with a particular choice of the graviphoton background $\epsilon_1 = -\epsilon_2$ is mentioned earlier in [4] and was used later in [10] to calculate the expectation value $\mathrm{tr}\langle\phi^2\rangle$.

As a further check, let us go to the limit when $T_1 \to 0$, $T_2 \to 0$. As we have explained above, one expects to find the generating function of Poincare polynomial for the Hilbert scheme of points on $\mathbb{C}^2$. An easy calculation yields

$$Z_{\text{adj}}|_{T_1,T_2=0} = \exp\sum_{n=1}^{\infty}\frac{\mathbf{q}^n}{n\left(1-T_m^n\mathbf{q}^n\right)} = \exp\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\mathbf{q}^{1+k}T_m^k\right)^n}{n} = \prod_{k=0}^{\infty}\frac{1}{1-T_m^k\mathbf{q}^{k+1}}, \tag{2.10}$$

which indeed after identifying T_m with Poincare parameter t^2 reproduces the well-known result (see e.g. [11]). Now let us go back to the general case. In various domains of the variables T_1, T_2 we can represent (2.8) as an infinite product as we did in the special case in (2.10). Let us consider separately the cases:

(a) $|T_1| < 1$, $|T_2| < 1$, $|T_m\mathbf{q}| < 1$

In this region (2.8) could be rewritten as

$$Z_{\text{adj}} = \exp\left\{\sum_{n=1}^{\infty}\sum_{k,i,j=0}^{\infty}\frac{\mathbf{q}^n}{n}T_1^{ni}T_2^{nj}(T_m\mathbf{q})^{nk}\left(1-T_m^nT_1^n\right)\left(1-T_m^nT_2^n\right)\right\}. \tag{2.11}$$

Performing summation over n we get

$$Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty}\frac{\left(1-\mathbf{q}^{k+1}T_m^{k+1}T_1^{i+1}T_2^j\right)\left(1-\mathbf{q}^{k+1}T_m^{k+1}T_1^iT_2^{j+1}\right)}{\left(1-\mathbf{q}^{k+1}T_m^kT_1^iT_2^j\right)\left(1-\mathbf{q}^{k+1}T_m^{k+2}T_1^{i+1}T_2^{j+1}\right)}. \tag{2.12}$$

(b) $|T_1| > 1, |T_2| < 1, |T_m\mathbf{q}| < 1$

In this region we expand (2.8) over $1/T_1$:

$$Z_{\text{adj}} = \exp\left\{\sum_{n=1}^{\infty}\sum_{k,i,j=0}^{\infty}\frac{-\mathbf{q}^n}{n}T_1^{-ni}T_2^{nj}(T_m\mathbf{q})^{nk}\left(1-T_m^nT_1^n\right)\left(1-T_m^nT_2^n\right)T_1^{-n}\right\}, \tag{2.13}$$

which leads to

$$Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty}\frac{\left(1-\mathbf{q}^{k+1}T_m^kT_1^{-i-1}T_2^j\right)\left(1-\mathbf{q}^{k+1}T_m^{k+2}T_1^{-i}T_2^{j+1}\right)}{\left(1-\mathbf{q}^{k+1}T_m^{k+1}T_1^{-i}T_2^j\right)\left(1-\mathbf{q}^{k+1}T_m^{k+1}T_1^{-i-1}T_2^{j+1}\right)}. \tag{2.14}$$

Recently Iqbal, Kozçaz and Shabir [6] have computed the partition function of these $U(1)$ adjoint theories using the refined topological vertex formalism [12]. And, since formula (2.8) was known to the present authors for quite a while, we performed detailed comparison with their results. To make contact with the formulae of Iqbal *et al* we need the following dictionary: $T_m = Q_m(t/q)^{1/2}$, $T_1 = 1/t$, $T_2 = q$, $\mathbf{q} = Q(q/t)^{1/2}$. In terms of these variables, equations (2.12) and (2.14) take the forms

(a) $|t| > 1, |q| < 1, |QQ_m| < 1$

$$Z_{\text{adj}} = \prod_{i,j,k=1}^{\infty}\frac{\left(1-Q^kQ_m^kq^{i-1}t^{-j}\right)\left(1-Q^kQ_m^kq^it^{1-j}\right)}{\left(1-Q^kQ_m^{k+1}q^{i-\frac{1}{2}}t^{-j+\frac{1}{2}}\right)\left(1-Q^kQ_m^{k-1}q^{i-\frac{1}{2}}t^{-j+\frac{1}{2}}\right)} \tag{2.15}$$

and

(b) $|t| < 1, |q| < 1, |QQ_m| < 1$

$$Z_{\text{adj}} = \prod_{i,j,k=1}^{\infty}\frac{\left(1-Q^kQ_m^{k+1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)\left(1-Q^kQ_m^{k-1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)}{\left(1-Q^kQ_m^kq^{i-1}t^{j-1}\right)\left(1-Q^kQ_m^kq^it^j\right)}. \tag{2.16}$$

These equations come rather close, but certainly do not coincide with those given in [6] at the end of the part 3.2. The reason for this discrepancy seems to us as follows. According to [6] the refined topological vertex method for the 5D $U(1)$ theory with adjoint matter leads to (see equation (4.6) of [6]; below we omit the 'perturbative part' $\prod_{i',j'=1}^{\infty}(1-Q_mq^{-\rho_{i'}}t^{-\rho_{j'}})$)

$$Z = \prod_{k=1}^{\infty}\left(1-Q^kQ_m^k\right)^{-1}\prod_{i,j=1}^{\infty}\left(1-Q^kQ_m^{k-1}q^{-\rho_i}t^{-\rho_j}\right)\left(1-Q^kQ_m^kq^{\rho_i-1/2}t^{-\rho_j+1/2}\right)$$
$$\times\left(1-Q^kQ_m^kq^{-\rho_i+1/2}t^{\rho_j-1/2}\right)\left(1-Q^kQ_m^{k+1}q^{\rho_i}t^{\rho_j}\right), \tag{2.17}$$

where $\rho_i = -i + 1/2$. But four factors under the product over i, j have different, excluding each other, regions of convergence. Thus this infinite product should be treated very carefully. Unfortunately, the authors of [6] do not tell what analytic continuation procedure they have adopted to pass from their equation (4.6) to those presented at the end of the part 3.2, but we will demonstrate now that one perhaps the simplest approach directly leads to our conjectural formula (2.8). We simply examine the product over each factor separately within its region of convergence and only after that continue analytically to a common region of the parameters. Thus, for the first factor in (2.17) we have

$$\prod_{k=1}^{\infty}\left(1-Q^kQ_m^k\right)^{-1}=\exp\sum_{n,k=1}^{\infty}\frac{(QQ_m)^{nk}}{n}=\exp\sum_{n=1}^{\infty}\frac{(QQ_m)^n}{n(1-(QQ_m)^n)}. \tag{2.18}$$

For the next factor (assuming $q<1, t<1$)

$$\begin{aligned}\prod_{k,i,j=1}^{\infty}\left(1-Q^kQ_m^{k-1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)&=\exp\sum_{n,k,i,j=1}^{\infty}\frac{-Q^{kn}Q_m^{(k-1)n}q^{(i-\frac{1}{2})n}t^{(j-\frac{1}{2})n}}{n}\\&=\exp\sum_{n=1}^{\infty}\frac{-Q^nq^{\frac{n}{2}}t^{\frac{n}{2}}}{(1-(QQ_m)^n)(1-q^n)(1-t^n)}.\end{aligned} \tag{2.19}$$

Similarly for $q>1, t<1$

$$\prod_{k,i,j=1}^{\infty}\left(1-Q^kQ_m^kq^{-i}t^j\right)=\exp\sum_{n=1}^{\infty}\frac{-Q^nQ_m^nq^{-n}t^n}{n(1-(QQ_m)^n)(1-q^{-n})(1-t^n)}, \tag{2.20}$$

for $q<1, t>1$

$$\prod_{k,i,j=1}^{\infty}\left(1-Q^kQ_m^kq^{i}t^{-j}\right)=\exp\sum_{n=1}^{\infty}\frac{-Q^nQ_m^nq^{n}t^{-n}}{n(1-(QQ_m)^n)(1-q^{n})(1-t^{-n})}, \tag{2.21}$$

and, finally for $q>1, t>1$

$$\prod_{k,i,j=1}^{\infty}\left(1-Q^kQ_m^{k+1}q^{-i+\frac{1}{2}}t^{-j+\frac{1}{2}}\right)=\exp\sum_{n=1}^{\infty}\frac{-Q^nQ_m^{2n}q^{-\frac{n}{2}}t^{-\frac{n}{2}}}{n(1-(QQ_m)^n)(1-q^{-n})(1-t^{-n})}. \tag{2.22}$$

Note that the rhs of above expressions are defined also outside of their initial convergence region. Combining all these together we get

$$Z=\exp\sum_{n=1}^{\infty}\frac{(QQ_m)^n\left(q^{\frac{n}{2}}t^{\frac{n}{2}}-Q_m^n\right)\left(q^{\frac{n}{2}}t^{\frac{n}{2}}-Q_m^{-n}\right)}{n(1-(QQ_m)^n)(1-q^n)(1-t^n)}, \tag{2.23}$$

which in terms of the parameters $\mathbf{q}$, T_1, T_2 exactly coincides with our conjectural result (2.8).

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